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Multiple orthogonal polynomials associated with Macdonald functions

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Abstract

We consider multiple orthogonal polynomials corresponding to two Macdonald functions (modified Bessel functions of the second kind), with emphasis on the polynomials on the diagonal of the Hermite-Padé table. We give some properties of these polynomials: differential properties, a Rodrigues type formula and explicit formulas for the third order linear recurrence relation.

1 Macdonald functions

In this paper we will investigate certain polynomials satisfying orthogonality properties with respect to weight functions related to Macdonald functions $K_\nu(z)$. These Macdonald functions are modified Bessel functions of the second kind and satisfy the differential equation

$$z^2u'' + zu' - (z^2+\nu^2)u = 0,$$

for which they are the solution that remains bounded as z tends to infinity on the real line. They can be given by the following integral representations

$$\begin{aligned} K_\nu(z) &= \left(\frac{\pi}{2z}\right)^{1/2} \frac{e^{-z}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-t} t^{\nu-1/2} (1+\frac{t}{2z})^{\nu-1/2}\, dt && (1.1) \\ &= \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \exp(-t-\frac{z^2}{4t}) t^{-\nu-1}\, dt && (1.2) \end{aligned}$$

(see, e.g., The Bateman Manuscript Project [2, Chapter VII], Nikiforov and Uvarov [5, pp. 223–226]). Useful properties are

$$\begin{aligned} -2K_\nu'(z) &= K_{\nu-1}(z) + K_{\nu+1}(z), && (1.3) \\ -\frac{2\nu}{z}K_\nu(z) &= K_{\nu-1}(z) - K_{\nu+1}(z). && (1.4) \end{aligned}$$

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These functions have the asymptotic behavior

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.5}$$

and near the origin

$$\begin{cases} z^\nu K_\nu(z) = 2^{\nu-1}\Gamma(\nu) + o(1) & \text{if } z \to 0 \\ K_0(z) = -\log z + O(1), & \text{if } z \to 0 \end{cases} \tag{1.6}$$

The moments of these functions are given explictly by

$$\int_0^\infty x^\mu K_\nu(x)\, dx = 2^{\mu-1}\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}), \qquad \Re(\mu \pm \nu + 1) > 0. \tag{1.7}$$

We will use as weight functions the scaled Macdonald functions

$$\rho_\nu(x) = 2x^{\nu/2} K_\nu(2\sqrt{x}), \qquad x > 0. \tag{1.8}$$

which behave like $\exp(-2\sqrt{x})$ as $x \to \infty$. From the properties (1.3) and (1.4) one easily obtains the differential properties

$$(x^{-\nu}\rho_\nu)' = -x^{-(\nu+1)}\rho_{\nu+1}, \tag{1.9}$$

$$\rho'_{\nu+1} = -\rho_\nu, \tag{1.10}$$

so that ρ_ν satisfies the second order linear differential equation $[x^{\nu+1}(x^{-\nu}\rho_\nu)']' = \rho_\nu$. By a simple change of variable in (1.7) we can find the moments of ρ_ν which are given by

$$\int_0^\infty x^\mu \rho_\nu(x)\, dx = \Gamma(\mu+\nu+1)\Gamma(\mu+1), \tag{1.11}$$

in particular we see that the nth moment of ρ_0 is given by $(n!)^2$.

In [8] A. P. Prudnikov formulated as an open problem the construction of the orthogonal polynomials associated with the weight function ρ_ν (and for more general weights, known as ultra-exponential weight functions). In the present paper we will show that it is more natural to investigate multiple orthogonal polynomials for two Macdonald weights ρ_ν and $\rho_{\nu+1}$ since for these multiple orthogonal polynomials one has nice differential properties, a Rodrigues formula, and an explicit recurrence relation.

2 Multiple orthogonal polynomials

We will investigate two types of multiple orthogonal polynomials for the system of weights $(\rho_\nu, \rho_{\nu+1})$ $(\nu \geq 0)$ on $[0, \infty)$ with an additional factor x^α. Let $n, m \in \mathbb{N}$ and $\alpha > -1$, then the *type 1 polynomials* $(A^\alpha_{n,m}, B^\alpha_{n,m})$ are such that $A^\alpha_{n,m}$ is of degree n and $B^\alpha_{n,m}$ is of degree m, and they satisfy the orthogonality conditions

$$\int_0^\infty [A^\alpha_{n,m}(x)\rho_\nu(x) + B^\alpha_{n,m}(x)\rho_{\nu+1}(x)]x^{k+\alpha}\, dx = 0, \qquad k = 0, 1, 2, \ldots, n+m. \tag{2.1}$$

This gives $n+m+1$ linear homogeneous equations for the $n+m+2$ unknown coefficients of the polynomials $A^\alpha_{n,m}$ and $B^\alpha_{n,m}$, so that we can find the type 1 polynomials up to a multiplicative factor, which we will fix later. It will be convenient to use the notation

$$q^\alpha_{n,m}(x) = A^\alpha_{n,m}(x)\rho_\nu(x) + B^\alpha_{n,m}(x)\rho_{\nu+1}(x).$$

Type 2 polynomials $p^\alpha_{n,m}$ are such that $p^\alpha_{n,m}$ is of degree $n+m$ and satisfies the multiple orthogonality conditions

$$\int_0^\infty p^\alpha_{n,m}(x)\rho_\nu(x)x^{k+\alpha}\,dx = 0 \qquad k = 0,1,2,\ldots,n-1, \tag{2.2}$$

$$\int_0^\infty p^\alpha_{n,m}(x)\rho_{\nu+1}(x)x^{k+\alpha}\,dx = 0 \qquad k = 0,1,2,\ldots,m-1. \tag{2.3}$$

This means that we distribute the $n+m$ orthogonality conditions over the two weights $x^\alpha\rho_\nu$ and $x^\alpha\rho_{\nu+1}$. This gives $n+m$ linear homogeneous equations for the $n+m+1$ unknown coefficients of $p^\alpha_{n,m}$. For type 2 polynomials we will consider monic polynomials, thereby fixing the leading coefficient to be 1 and leaving $n+m$ coefficients to be determined from (2.2)–(2.3).

Multiple orthogonal polynomials are related to Hermite-Padé simultaneous rational approximation of a system of Markov functions near infinity, (see, e.g., Nikishin and Sorokin [6] and Aptekarev [1]). In the present situation the functions to be approximated are

$$f_1(z) = \int_0^\infty \frac{x^\alpha\rho_\nu(x)}{z-x}\,dx, \quad f_2(z) = \int_0^\infty \frac{x^\alpha\rho_{\nu+1}(x)}{z-x}\,dx.$$

Type 1 Hermite-Padé approximation (Latin type) consists of finding polynomials $A_{n,m}$, $B_{n,m}$, and $C_{n,m}$ such that

$$A_{n,m}(z)f_1(z) + B_{n,m}(z)f_2(z) - C_{n,m}(z) = O(z^{-n-m-2}) \qquad z\to\infty,$$

and type 2 Hermite-Padé approximation (German type) is simultaneous rational approximation of (f_1,f_2) with a common denominator $p_{n,m}$, i.e., one wants polynomials $p_{n,m}$, $R_{n,m}$ and $S_{n,m}$ such that

$$\begin{aligned} p_{n,m}(z)f_1(z) - R_{n,m}(z) &= O(z^{-n-1}), &\qquad z\to\infty,\\ p_{n,m}(z)f_2(z) - S_{n,m}(z) &= O(z^{-m-1}), &\qquad z\to\infty. \end{aligned}$$

The polynomials $A_{n,m}, B_{n,m}$ then are precisely the type 1 polynomials $A^\alpha_{n,m}$ and $B^\alpha_{n,m}$ and the polynomial $p_{n,m}$ is the type 2 polynomial $p^\alpha_{n,m}$. The numerator $C_{n,m}$ is then given by

$$C_{n,m}(z) = \int_0^\infty \left[\frac{A^\alpha_{n,m}(z)-A^\alpha_{n,m}(x)}{z-x}\rho_\nu(x) + \frac{B^\alpha_{n,m}(z)-B^\alpha_{n,m}(x)}{z-x}\rho_{\nu+1}(x)\right]x^\alpha\,dx,$$

and for type 2 approximation the numerators are

$$\begin{aligned} R_{n,m}(z) &= \int_0^\infty \frac{p^\alpha_{n,m}(z)-p^\alpha_{n,m}(x)}{z-x}\rho_\nu(x)x^\alpha\,dx,\\ S_{n,m}(z) &= \int_0^\infty \frac{p^\alpha_{n,m}(z)-p^\alpha_{n,m}(x)}{z-x}\rho_{\nu+1}(x)x^\alpha\,dx. \end{aligned}$$

Observe that we can write the system (f_1, f_2) as

$$f_1(z) = \int_0^\infty f(x)\frac{\rho_{\nu+1}(x)x^\alpha}{z-x}\,dx, \quad f_2(z) = \int_0^\infty \frac{\rho_{\nu+1}(x)x^\alpha}{z-x}\,dx,$$

where f is itself a Markov function

$$f(x) = \frac{\rho_\nu(x)}{\rho_{\nu+1}(x)} = \frac{1}{\pi^2}\int_0^\infty \frac{s^{-1}\,ds}{(x+s)[J_{\nu+1}^2(2\sqrt{s}) + Y_{\nu+1}^2(2\sqrt{s})]},$$

which follows from a result by Ismail [3]. Note however that f is a Markov function of a positive measure for which not all the moments exist. Nevertheless we can still say that (f_1, f_2) is a *Nikishin system*, which guarantees that the polynomials $A_{n,m}^\alpha$, $B_{n,m}^\alpha$ and $p_{n,m}^\alpha$ all can be computed and their degrees are exactly n, m, and $n+m$ respectively. Furthermore the zeros of $q_{n,n}^\alpha$, $q_{n+1,n}^\alpha$, $p_{n,n}^\alpha$ and $p_{n+1,n}^\alpha$ will all be on $(0,\infty)$.

3 Differential properties

In this section we will give some differential properties for the type 1 and type 2 multiple orthogonal polynomials. We will only consider the multiple orthogonal polynomials on the diagonal or close to the diagonal, i.e., when $n = m$ or $n = m+1$. These are the most natural and they satisfy interesting relations.

Theorem 1 *For the type 2 multiple orthogonal polynomials we have for every $\alpha > -1$*

$$\frac{d}{dx}p_{n,n}^\alpha(x) = 2np_{n,n-1}^{\alpha+1}(x), \quad \frac{d}{dx}p_{n,n-1}^\alpha(x) = (2n-1)p_{n-1,n-1}^{\alpha+1}(x). \tag{3.1}$$

This means that the differential operator acting on type 2 multiple orthogonal polynomials lowers the degree by one and raises the α by one. This should be compared with the corresponding differential property

$$\frac{d}{dx}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x)$$

for Laguerre polynomials (see, e.g., Szegő [7, Eq. (5.1.14) on p. 102]). The normalizing constant is different here since Laguerre polynomials are not monic polynomials.

Proof: We begin by using (2.2)

$$\int_0^\infty p_{n,n}^\alpha(x)\rho_\nu(x)x^{k+\alpha}\,dx = 0, \qquad k = 0,1,\ldots,n-1,$$

and then use (1.10) to find

$$-\int_0^\infty p_{n,n}^\alpha(x)\rho_{\nu+1}'(x)x^{k+\alpha}\,dx = 0, \qquad k = 0,1,\ldots,n-1.$$

Integration by parts then gives

$$-p_{n,n}^\alpha(x)\rho_{nu+1}(x)x^{k+\alpha}\Big|_0^\infty + \int_0^\infty \left[p_{n,n}^\alpha(x)x^{k+\alpha}\right]'\rho_{\nu+1}(x)\,dx = 0, \qquad k = 0,1,\ldots,n-1.$$

The integrated terms vanish for $k \geq 1$ and $\alpha > -1$ because of the behavior of $\rho_{\nu+1}$ near 0 and ∞, given in (1.5) and (1.6). Now work out the differentiation of the product, then we have

$$\int_0^\infty [p_{n,n}^\alpha(x)]' x^{k+\alpha} \rho_{\nu+1}(x)\, dx \\ + \; (k+\alpha) \int_0^\infty p_{n,n}^\alpha(x) x^{k+\alpha-1} \rho_{\nu+1}(x)\, dx = 0, \qquad k = 1,2,\ldots,n-1.$$

The second integral is zero for $k = 1, 2, \ldots, n$ because of the orthogonality condition (2.3), hence we conclude that

$$\int_0^\infty [p_{n,n}^\alpha(x)]' x^{k+\alpha} \rho_{\nu+1}(x)\, dx = 0, \qquad k = 1,2,\ldots,n-1,$$

or equivalently

$$\int_0^\infty [p_{n,n}^\alpha(x)]' x^{\ell+\alpha+1} \rho_{\nu+1}(x)\, dx = 0, \qquad \ell = 0,1,\ldots,n-2. \tag{3.2}$$

Next we do a similar analysis with (2.3), in which we use (1.9) to find

$$-\int_0^\infty p_{n,n}^\alpha(x) x^{k+\alpha+\nu+1} [x^{-\nu} \rho_\nu(x)]'\, dx = 0, \qquad k = 0,1,\ldots,n-1.$$

Integration by parts gives

$$-p_{n,n}^\alpha(x) x^{k+\alpha+1} \rho_\nu(x) \Big|_0^\infty \\ + \int_0^\infty \left[p_{n,n}^\alpha(x) x^{k+\alpha+\nu+1}\right]' x^{-\nu} \rho_\nu(x)\, dx = 0, \qquad k = 0,1,\ldots,n-1.$$

The integrated terms vanish for $k \geq 0$ and $\alpha > -1$. Working out the derivative of the product then gives

$$\int_0^\infty [p_{n,n}^\alpha(x)]' x^{k+\alpha+1} \rho_\nu(x)\, dx \\ + \; (k+\alpha+\nu+1) \int_0^\infty p_{n,n}^\alpha(x) x^{k+\alpha} \rho_\nu(x)\, dx = 0, \qquad k = 0,1,\ldots,n-1.$$

The second integral is zero for $k = 0, 1, \ldots, n-1$ because of the orthogonality (2.2), hence we have

$$\int_0^\infty [p_{n,n}^\alpha(x)]' x^{k+\alpha+1} \rho_\nu(x)\, dx = 0, \qquad k = 0,1,\ldots,n-1. \tag{3.3}$$

Now $[p_{n,n}^\alpha]'$ is a polynomial of degree $2n-1$ with leading coefficient $2n$ (since we normalized the type 2 multiple orthogonal polynomials by taking monic polynomials), and by (3.2) and (3.3) it satisfies the orthogonality conditions (2.2)–(2.3) for the type 2 multiple orthogonal polynomial $p_{n,n-1}^{\alpha+1}$. By unicity we therefore have $[p_{n,n}^\alpha(x)]' = 2n p_{n,n-1}^{\alpha+1}(x)$. A similar reasoning also gives the result $[p_{n,n-1}^\alpha(x)]' = (2n-1) p_{n-1,n-1}^{\alpha+1}(x)$. Note however that the analysis does not work when $m \notin \{n, n-1\}$. ■

There is a similar differential property for type 1 multiple orthogonal polynomials which complements the differential property of the type 2 multiple orthogonal polynomials given in the previous theorem.

Theorem 2 *For the type 1 multiple orthogonal polynomials we have for every $\alpha > 0$*

$$\frac{d}{dx}[x^\alpha q_{n,n}^\alpha(x)] = x^{\alpha-1} q_{n+1,n}^{\alpha-1}(x), \quad \frac{d}{dx}[x^\alpha q_{n,n-1}^\alpha(x)] = x^{\alpha-1} q_{n,n}^{\alpha-1}(x). \tag{3.4}$$

This means that the differential operator acting on x^α times the type 1 polynomials raises the degree by one and lowers the α by one. This should be compared with the corresponding differential property

$$\frac{d}{dx}[e^{-x} x^\alpha L_n^\alpha(x)] = (n+1) e^{-x} x^{\alpha-1} L_{n+1}^{\alpha-1}(x)$$

for Laguerre polynomials. What we will really prove is that the derivative of $x^\alpha q_{n,n}^\alpha(x)$ is proportional to $x^{\alpha-1} q_{n+1,n}^{\alpha-1}(x)$ (and similarly for the derivative of $x^\alpha q_{n,n-1}^\alpha(x)$ which is proportional to $x^{\alpha-1} q_{n,n}^{\alpha-1}(x)$). The choice of the proportionality factor one in (3.4) will fix the normalization which was left unspecified by the homogeneous system of equations (2.1).

Proof: We begin by using (2.1), which we can write as

$$\int_0^\infty x^\alpha q_{n,n}^\alpha (x^{k+1})' \, dx = 0, \qquad k = 0, 1, \ldots, 2n.$$

Integration by parts then gives

$$q_{n,n}^\alpha(x) x^{k+\alpha+1}\Big|_0^\infty - \int_0^\infty [x^\alpha q_{n,n}^\alpha(x)]' x^{k+1} \, dx = 0, \qquad k = 0, 1, \ldots, 2n.$$

The integrated terms will vanish for every $\alpha > -1$ and $k \geq 0$. For the integrand of the integral we have

$$\begin{aligned}
[x^\alpha q_{n,n}^\alpha(x)]' &= [x^\alpha A_{n,n}^\alpha(x) \rho_\nu(x)]' + [x^\alpha B_{n,n}^\alpha(x) \rho_{\nu+1}(x)]' \\
&= \alpha x^{\alpha-1} [A_{n,n}^\alpha \rho_\nu(x) + B_{n,n}^\alpha(x) \rho_{\nu+1}(x)] \\
&\quad + x^\alpha [(A_{n,n}^\alpha(x))' \rho_\nu(x) + A_{n,n}^\alpha(x) \rho_\nu'(x) \\
&\quad + (B_{n,n}^\alpha(x))' \rho_{\nu+1}(x) + B_{n,n}(x) \rho_{\nu+1}'(x)].
\end{aligned}$$

Now use (1.9)–(1.10) to find

$$\begin{aligned}
[x^\alpha q_{n,n}^\alpha(x)]' &= x^{\alpha-1} [((\alpha+\nu) A_{n,n}^\alpha(x) + x(A_{n,n}^\alpha(x))' - x B_{n,n}^\alpha(x)) \rho_\nu(x) \\
&\quad + (\alpha B_{n,n}^\alpha(x) - A_{n,n}^\alpha(x) + x(B_{n,n}^\alpha(x))') \rho_{\nu+1}(x)].
\end{aligned}$$

Observe that $(\alpha+\nu) A_{n,n}^\alpha(x) + x(A_{n,n}^\alpha(x))' - x B_{n,n}^\alpha(x)$ is a polynomial of degree at most $n+1$ and $\alpha B_{n,n}^\alpha(x) - A_{n,n}^\alpha(x) + x(B_{n,n}^\alpha(x))'$ is a polynomial of degree at most n, so that we can write

$$[x^\alpha q_{n,n}^\alpha(x)]' = x^{\alpha-1} [P_{n+1}(x) \rho_\nu(x) + Q_n(x) \rho_{\nu+1}(x)] \tag{3.5}$$

with polynomials P_{n+1} and Q_n of degree at most $n+1$ and n respectively. We therefore have

$$\int_0^\infty x^{\alpha-1} [P_{n+1}(x) \rho_\nu(x) + Q_n(x) \rho_{\nu+1}(x)] x^{k+1} \, dx = 0, \qquad k = 0, 1, \ldots, 2n.$$

In addition to this, we also have

$$\int_0^\infty [x^\alpha q_{n,n}^\alpha(x)]'\,dx = x^\alpha q_{n,n}^\alpha(x)\Big|_0^\infty = 0,$$

so that

$$\int_0^\infty [x^\alpha q_{n,n}^\alpha(x)]' x^\ell\,dx = 0, \qquad \ell = 0,1,\ldots,2n+1.$$

In view of (2.1) and (3.5) this means that

$$[x^\alpha q_{n,n}^\alpha(x)]' = \text{constant } x^{\alpha-1} q_{n+1,1}^{\alpha-1}(x).$$

Since $q_{n,n}^\alpha$ and $q_{n+1,n}^{\alpha-1}$ are only determined up to a normalizing factor, we can choose the constant equal to one, thereby fixing the normalization.

We can repeat the analysis for $q_{n,n-1}^\alpha$ with minor changes. In this case we have

$$\begin{aligned} [x^\alpha q_{n,n-1}^\alpha(x)]' &= x^{\alpha-1}[((\alpha+\nu)A_{n,n-1}^\alpha(x) + x(A_{n,n-1}^\alpha(x))' - xB_{n,n-1}^\alpha(x))\rho_\nu(x) \\ &\quad + (\alpha B_{n,n-1}^\alpha(x) - A_{n,n-1}^\alpha(x) + x(B_{n,n-1}^\alpha(x))')\rho_{\nu+1}(x)]. \end{aligned}$$

and since $(\alpha+\nu)A_{n,n-1}^\alpha(x) + x(A_{n,n-1}^\alpha(x))' - xB_{n,n-1}^\alpha(x)$ is a polynomial of degree at most n and $\alpha B_{n,n-1}^\alpha(x) - A_{n,n-1}^\alpha(x) + x(B_{n,n-1}^\alpha(x))'$ a polynomial of degree at most n also, the orthogonality

$$\int_0^\infty [x^\alpha q_{n,n-1}^\alpha(x)]' x^\ell\,dx = 0, \qquad \ell = 0,1,\ldots,2n$$

gives the required result.

Observe that the reasoning does not work for $[x^\alpha q_{n,m}^\alpha(x)]'$ when $m \notin \{n, n-1\}$. ■

The proof also shows that

$$\begin{aligned} A_{n+1,n}^{\alpha-1}(x) &= (\alpha+\nu)A_{n,n}^\alpha(x) + x[A_{n,n}^\alpha(x)]' - xB_{n,n}^\alpha(x), \\ B_{n+1,n}^{\alpha-1}(x) &= \alpha B_{n,n}^\alpha(x) - A_{n,n}^\alpha(x) + x[B_{n,n}^\alpha(x)]', \end{aligned}$$

and similarly

$$\begin{aligned} A_{n,n}^{\alpha-1}(x) &= (\alpha+\nu)A_{n,n-1}^\alpha(x) + x[A_{n,n-1}^\alpha(x)]' - xB_{n,n-1}^\alpha(x), \\ B_{n,n}^{\alpha-1}(x) &= \alpha B_{n,n-1}^\alpha(x) - A_{n,n-1}^\alpha(x) + x[B_{n,n-1}^\alpha(x)]'. \end{aligned}$$

4 Rodrigues formula

As a consequence of Theorem 2 we have the following Rodrigues formula for the type 1 multiple orthogonal polynomials.

Theorem 3 *The type 1 multiple orthogonal polynomials can be obtained from*

$$\frac{d^{2n}}{dx^{2n}}\left(x^{2n+\alpha}\rho_\nu(x)\right) = x^\alpha q_{n,n-1}^\alpha(x), \quad \frac{d^{2n+1}}{dx^{2n+1}}\left(x^{2n+1+\alpha}\rho_\nu(x)\right) = x^\alpha q_{n,n}^\alpha(x). \tag{4.1}$$

Proof: By combining the two formulas in (3.4) we get

$$\frac{d^2}{dx^2}\left(x^\alpha q_{n,n-1}^\alpha(x)\right) = x^{\alpha-2} q_{n+1,n}^{\alpha-2}.$$

Iterate this k times to get

$$\frac{d^{2k}}{dx^{2k}}\left(x^\alpha q_{n,n-1}^\alpha(x)\right) = x^{\alpha-2k} q_{n+k,n+k-1}^{\alpha-2k}.$$

Choose $n=0$ and $\alpha = 2k+\beta$ to find

$$\frac{d^{2k}}{dx^{2k}}\left(x^{2k+\beta} q_{0,-1}^{2k+\beta}(x)\right) = x^{\beta} q_{k,k-1}^{\beta}.$$

Now $q_{0,-1}^\alpha(x) = A_{0,-1}^\alpha \rho_\nu(x)$, where $A_{0,-1}^\alpha$ is a constant, which we will take equal to one to fix normalization. Thus, if we replace β by α and k by n, then we find the first formula in (4.1).

Similarly, we can use

$$\frac{d}{dx}\left(x^\alpha q_{n,n-1}^\alpha(x)\right) = x^{\alpha-1} q_{n,n}^{\alpha-1}(x),$$

and differentiate it $2k$ times, to find

$$\frac{d^{2k+1}}{dx^{2k+1}}\left(x^\alpha q_{n,n-1}^\alpha(x)\right) = x^{\alpha-2k-1} q_{n+k,n+k}^{\alpha-2k-1}(x).$$

Choose $n=0$ and $\alpha = 2k+1+\beta$ to find

$$\frac{d^{2k+1}}{dx^{2k+1}}\left(x^{2k+1+\beta} \rho_\nu(x)\right) = x^{\beta} q_{k,k}^{\beta}(x).$$

Replacing β by α and k by n then gives the required formula. ■

The Rodrigues formula allows us to compute the type 1 multiple orthogonal polynomials explicitly. There is a relationship between type 1 and type 2 polynomials (which is not typical for the Macdonald weights but holds in general) and this allows us to compute the type 2 multiple polynomials as well. Indeed, we can write

$$\begin{pmatrix} q_{n,n}^\alpha \\ q_{n,n-1}^\alpha \end{pmatrix} = \begin{pmatrix} A_{n,n}^\alpha & B_{n,n}^\alpha \\ A_{n,n-1}^\alpha & B_{n,n-1}^\alpha \end{pmatrix} \begin{pmatrix} \rho_\nu \\ \rho_{\nu+1} \end{pmatrix},$$

so that we have

$$\begin{pmatrix} A_{n,n}^\alpha & B_{n,n}^\alpha \\ A_{n,n-1}^\alpha & B_{n,n-1}^\alpha \end{pmatrix}^{-1} \begin{pmatrix} q_{n,n}^\alpha \\ q_{n,n-1}^\alpha \end{pmatrix} = \begin{pmatrix} \rho_\nu \\ \rho_{\nu+1} \end{pmatrix}.$$

Writing the inverse of a matrix as the adjoint matrix divided by the determinant gives

$$\begin{pmatrix} B_{n,n-1}^\alpha & -B_{n,n}^\alpha \\ -A_{n,n-1}^\alpha & A_{n,n}^\alpha \end{pmatrix} \begin{pmatrix} q_{n,n}^\alpha \\ q_{n,n-1}^\alpha \end{pmatrix} = [A_{n,n}^\alpha B_{n,n-1}^\alpha - A_{n,n-1}^\alpha B_{n,n}^\alpha] \begin{pmatrix} \rho_\nu \\ \rho_{\nu+1} \end{pmatrix}. \tag{4.2}$$

The polynomial $A^\alpha_{n,n}B^\alpha_{n,n-1} - A^\alpha_{n,n-1}B^\alpha_{n,n}$ is of degree at most $2n$ and satisfies

$$\int_0^\infty [A^\alpha_{n,n}(x)B^\alpha_{n,n-1}(x) - A^\alpha_{n,n-1}(x)B^\alpha_{n,n}(x)]\rho_\nu(x)x^{k+\alpha}\,dx$$
$$= \int_0^\infty [B^\alpha_{n,n-1}(x)q^\alpha_{n,n}(x) - B^\alpha_{n,n}(x)q^\alpha_{n,n-1}(x)]x^{k+\alpha}\,dx = 0, \qquad k = 0,1,\dots,n-1,$$

where we have used (4.2) and (2.1). Furthermore we also have

$$\int_0^\infty [A^\alpha_{n,n}(x)B^\alpha_{n,n-1}(x) - A^\alpha_{n,n-1}(x)B^\alpha_{n,n}(x)]\rho_{\nu+1}(x)x^{k+\alpha}\,dx$$
$$= \int_0^\infty [-A^\alpha_{n,n-1}(x)q^\alpha_{n,n}(x) + A^\alpha_{n,n}(x)q^\alpha_{n,n-1}(x)]x^{k+\alpha}\,dx = 0, \qquad k = 0,1,\dots,n-1.$$

Hence we can conclude that

$$A^\alpha_{n,n}(x)B^\alpha_{n,n-1}(x) - A^\alpha_{n,n-1}(x)B^\alpha_{n,n}(x) = \text{constant } p^\alpha_{n,n}(x),$$

where the constant is the leading coefficient of the polynomial. A similar reasoning using

$$\begin{pmatrix} q^\alpha_{n+1,n} \\ q^\alpha_{n,n} \end{pmatrix} = \begin{pmatrix} A^\alpha_{n+1,n} & B^\alpha_{n+1,n} \\ A^\alpha_{n,n} & B^\alpha_{n,n} \end{pmatrix} \begin{pmatrix} \rho_\nu \\ \rho_{\nu+1} \end{pmatrix},$$

gives

$$A^\alpha_{n+1,n}(x)B^\alpha_{n,n}(x) - A^\alpha_{n,n}(x)B^\alpha_{n+1,n}(x) = \text{constant } p^\alpha_{n+1,n}(x).$$

5 Recurrence relation

To simplify the notation we put

$$P_{2n}(x) = p^\alpha_{n,n}(x), \quad P_{2n+1}(x) = p^\alpha_{n+1,n}(x).$$

It is known that the sequence $P_n(x)$, $n = 0,1,2,\dots$ satisfies a third order recurrence relation of the form

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x) + d_nP_{n-2}(x), \tag{5.1}$$

(see, e.g., [4] or [6]). Explicit formulas for the recurrence coefficients are given in the following theorem.

Theorem 4 *The recurrence coefficients in (5.1) are given by*

$$\begin{aligned} b_n &= (n+\alpha+1)(3n+\alpha+2\nu) - (\alpha+1)(\nu-1) \\ c_n &= n(n+\alpha)(n+\alpha+\nu)(3n+2\alpha+\nu) \\ d_n &= n(n-1)(n+\alpha-1)(n+\alpha)(n+\alpha+\nu-1)(n+\alpha+\nu). \end{aligned}$$

Proof: We begin with the recurrence coefficients of even index, which are used in the recurrence relation

$$xp_{n,n}^{\alpha}(x) = p_{n+1,n}^{\alpha}(x) + b_{2n}p_{n,n}^{\alpha}(x) + c_{2n}p_{n,n-1}^{\alpha}(a) + d_{2n}p_{n-1,n-1}^{\alpha}(x). \tag{5.2}$$

We will write the polynomials $p_{n,m}^{\alpha}$ explicitly as

$$p_{n,m}^{\alpha}(x) = \sum_{k=0}^{n+m} a_{n,m}^{\alpha}(k)x^{n+m-k},$$

and since we are dealing with monic polynomials, we have

$$a_{n,m}^{\alpha}(0) = 1.$$

Comparing the coefficient of x^{2n} in (5.2) gives

$$b_{2n} = a_{n,n}^{\alpha}(1) - a_{n+1,n}^{\alpha}(1), \tag{5.3}$$

hence we need to know the coefficients $a_{n,m}^{\alpha}(1)$. Comparing the coefficient of x^{2n-2} and x^{2n-3} respectively in (3.1) gives the recurrence

$$(2n-1)a_{n,n}^{\alpha}(1) = 2na_{n,n-1}^{\alpha+1}(1), \quad (2n-2)a_{n,n-1}^{\alpha}(1) = (2n-1)a_{n-1,n-1}^{\alpha+1}(1).$$

Combining these relations gives

$$a_{n,n}^{\alpha}(1) = \frac{n}{n-1}a_{n-1,n-1}^{\alpha+2}(1),$$

which leads to

$$a_{n,n}^{\alpha}(1) = na_{1,1}^{\alpha+2n-2}(1).$$

In order to obtain an explicit formula, we compute $p_{1,1}^{\alpha}$ explicitly by solving the system of equations

$$\begin{aligned} \int_0^\infty (x^2 + a_{1,1}^{\alpha}(1)x + a_{1,1}^{\alpha}(2))x^{\alpha}\rho_{\nu}(x)\,dx &= 0 \\ \int_0^\infty (x^2 + a_{1,1}^{\alpha}(1)x + a_{1,1}^{\alpha}(2))x^{\alpha}\rho_{\nu+1}(x)\,dx &= 0. \end{aligned}$$

Using the moments (1.11) this gives

$$a_{1,1}^{\alpha}(1) = -2(2+\alpha)(2+\alpha+\nu), \tag{5.4}$$
$$a_{1,1}^{\alpha}(2) = (1+\alpha)(2+\alpha)(1+\alpha+\nu)(2+\alpha+\nu). \tag{5.5}$$

This gives

$$a_{n,n}^{\alpha}(1) = -2n(\alpha+2n)(\alpha+2n+\nu), \tag{5.6}$$

and since $2na_{n+1,n}^{\alpha}(1) = (2n+1)a_{n,n}^{\alpha+1}(1)$ this also gives

$$a_{n+1,n}^{\alpha}(1) = -(2n+1)(\alpha+2n+1)(\alpha+2n+\nu+1). \tag{5.7}$$

Inserting this in (5.3) gives the requested formula for the recurrence coefficient b_{2n}.

Next we compare the coefficient of x^{2n-1} in (5.2) to find

$$c_{2n} = a^{\alpha}_{n,n}(2) - a^{\alpha}_{n+1,n}(2) - b_{2n} a^{\alpha}_{n,n}(1). \tag{5.8}$$

This means that we also need to know $a^{\alpha}_{n,n}(2)$ and $a^{\alpha}_{n+1,n}(2)$. Compare the coefficient of x^{2n-3} and x^{2n-4} respectively in (3.1), then

$$(2n-2)a^{\alpha}_{n,n}(2) = 2n a^{\alpha+1}_{n,n-1}(2), \quad (2n-3)a^{\alpha}_{n,n-1}(2) = (2n-1)a^{\alpha+1}_{n-1,n-1}(2),$$

which combined gives

$$a^{\alpha}_{n,n}(2) = \frac{(2n)(2n-1)}{(2n-2)(2n-3)} a^{\alpha+2}_{n-1,n-1}(2) = \frac{(2n)(2n-1)}{2} a^{\alpha+2n-2}_{1,1}(2).$$

Using (5.5) this gives

$$a^{\alpha}_{n,n}(2) = n(2n-1)(\alpha+2n-1)(\alpha+2n)(\alpha+2n+\nu-1)(\alpha+2n+\nu), \tag{5.9}$$

and since $(2n-1)a^{\alpha}_{n+1,n}(2) = (2n+1)a^{\alpha+1}_{n,n}(2)$ this also gives

$$a^{\alpha}_{n+1,n}(2) = n(2n+1)(\alpha+2n)(\alpha+2n+1)(\alpha+2n+\nu)(\alpha+2n+\nu+1). \tag{5.10}$$

Inserting (5.9), (5.10), (5.6) and the expression for b_{2n} into (5.8) gives, after some straightforward calculus (or by using Maple) the requested expression for c_{2n}.

Finally, compare the coefficient of x^{2n-2} in (5.2), then

$$d_{2n} = a^{\alpha}_{n,n}(3) - a^{\alpha}_{n+1,n}(3) - b_{2n} a^{\alpha}_{n,n}(2) - c_{2n} a^{\alpha}_{n,n-1}(1), \tag{5.11}$$

so that we need $a^{\alpha}_{n,n}(3)$ and $a^{\alpha}_{n+1,n}(3)$. To this end we compare the coefficient of x^{2n-4} and x^{2n-5} respectively in (3.1) to find

$$(2n-3)a^{\alpha}_{n,n}(3) = 2n a^{\alpha}_{n,n-1}(3), \quad (2n-4)a^{\alpha}_{n,n-1}(3) = (2n-1)a^{\alpha}_{n-1,n-1}(3),$$

which combined gives

$$a^{\alpha}_{n+1,n}(3) = \frac{2n(2n+1)}{(2n-2)(2n-3)} a^{\alpha+2}_{n,n-1}(3) = \frac{(2n+1)(2n)(2n-1)}{6} a^{\alpha+2n-2}_{2,1}(3).$$

In order to find $a^{\alpha}_{2,1}(3)$ we will compute $p^{\alpha}_{2,1}$ explicitly by solving the system of equations

$$\begin{aligned}
\int_0^\infty (x^3 + a^{\alpha}_{2,1}(1)x^2 + a^{\alpha}_{2,1}(2)x + a^{\alpha}_{2,1}(3))x^{\alpha}\rho_\nu(x)\,dx &= 0\\
\int_0^\infty (x^3 + a^{\alpha}_{2,1}(1)x^2 + a^{\alpha}_{2,1}(2)x + a^{\alpha}_{2,1}(3))x^{\alpha+1}\rho_\nu(x)\,dx &= 0\\
\int_0^\infty (x^3 + a^{\alpha}_{2,1}(1)x^2 + a^{\alpha}_{2,1}(2)x + a^{\alpha}_{2,1}(3))x^{\alpha}\rho_{\nu+1}(x)\,dx &= 0
\end{aligned}$$

which by using (1.11) and some calculus gives

$$a^{\alpha}_{2,1}(3) = -(3+\alpha+\nu)(2+\alpha+\nu)(1+\alpha+\nu)(3+\alpha)(2+\alpha)(1+\alpha). \tag{5.12}$$

Using this gives

$$a_{n+1,n}^{\alpha}(3) = -\frac{(2n+1)(2n)(2n-1)}{6}$$
$$(\alpha+2n+\nu+1)(\alpha+2n+\nu)(\alpha+2n+\nu-1)(\alpha+2n+1)(\alpha+2n)(\alpha+2n-1), \quad (5.13)$$

and since $(2n-3)a_{n,n}^{\alpha}(3) = 2na_{n,n-1}^{\alpha+1}(3)$ we also have

$$a_{n,n}^{\alpha}(3) = -\frac{2n(2n-1)(2n-2)}{6}$$
$$(\alpha+2n+\nu)(\alpha+2n+\nu-1)(\alpha+2n+\nu-2)(\alpha+2n)(\alpha+2n-1)(\alpha+2n-2). \quad (5.14)$$

Using (5.14), (5.13), (5.9), (5.7) and the formulas for b_{2n} and c_{2n} in (5.11) then gives the requested expression for d_{2n}.

In a similar way we proceed with the odd indices which appear in the recurrence relation

$$xp_{n+1,n}^{\alpha}(x) = p_{n+1,n+1}^{\alpha}(x) + b_{2n+1}p_{n+1,n}^{\alpha}(x) + c_{2n+1}p_{n,n}^{\alpha}(x) + d_{2n+1}p_{n,n-1}^{\alpha}(x). \quad (5.15)$$

By comparing the coefficients of x^{2n+1}, x^{2n} and x^{2n-1} we get expressions for b_{2n+1}, c_{2n+1} and d_{2n+1} respectively in terms of the coefficients $a_{n,m}^{\alpha}$ and after working out these expressions we get the required formulas. ■

Observe that the recurrence coefficients have the asymptotic behavior

$$b_n \sim 3n^2, \quad c_n \sim 3n^4, \quad c_n \sim n^6. \quad (5.16)$$

References

[1] A. I. Aptekarev, *Multiple orthogonal polynomials*, J. Comput. Appl. Math. **99** (1998), 423–447.

[2] A. Erdélyi, *Higher Transcendental Functions*, Volume II, McGraw-Hill, New York, 1953

[3] M. E. H. Ismail, *Bessel functions and the infinite divisibility of the student t-distribution*, Ann. Prob. **5** (1977), 582–585

[4] V. Kaliaguine, A. Ronveaux, *On a system of "classical" polynomials of simultaneous orthogonality*, J. Comput. Appl. Math. **67** (1996), 207–217

[5] A. F. Nikiforov, V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Basel, 1988

[6] E. M. Nikishin, V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs **92**, Amer. Math. Soc., Providence, RI, 1991

[7] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. **23** (4th edition), 1975

[8] W. Van Assche (editor), *Open Problems, problem 9 by A. P. Prudnikov: orthogonal polynomials with ultra-exponential weight function*, J. Comput. Appl. Math. **48** (1993), 239–241.

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